LNAI 8234

Vicenç Torra
Yasuo Narukawa
Guillermo Navarro-Arribas
David Megías (Eds.)

Modeling Decisions for Artificial Intelligence

10th International Conference, MDAI 2013
Barcelona, Spain, November 2013
Proceedings

Springer

Some New Domain Restrictions in Social Choice, and Their Consequences

Salvador Barberà[1], Dolors Berga[2], and Bernardo Moreno[3]

[1] MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE, Departament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain
salvador.barbera@uab.cat

[2] Departament d'Economia, C/ Universitat de Girona 10, Universitat de Girona, 17071 Girona, Spain
dolors.berga@udg.edu

[3] Departamento de Teoría e Historia Económica, Facultad de Ciencias Económicas y Empresariales, Campus de El Ejido, 29071 Málaga, Spain
bernardo@uma.es

Abstract. Restricting the domains of definition of social choice functions is a classical method to test the robustness of impossibility results and to find conditions under which attractive methods to reach collective decisions can be identified, satisfying different sets of desirable properties. We survey a number of domains that we have recently explored, and exhibit results emerging for functions defined on them. In particular, we have identified a condition called top monotonicity under which the core of voting rules is non-empty, a second one called sequential inclusion where individual and group strategy-proofness become equivalent, and still a third condition called intertwinedness where the strategy-proofness of social choice functions is guaranteed as soon as they satisfy very simple monotonicity and invariance requirements.

Keywords: Strategy-proofness, group strategy-proofness, single-peaked preferences, separable preferences, top monotonicity, sequential inclusion, reshuffling invariance, monotonicity, intertwined domains.

1 Introduction

Social choice theory has a tradition of delivering impossibility results, but these are starting points for further, more constructive work. Indeed, impossibility results usually open the gate toward further understanding of design issues that eventually lead to positive proposals and to the characterization of rich classes of collective decision-making procedures. The crucial difference between positive and negative results is in most cases related to the domain of definition of the social choice rules under consideration. Arrow's impossibility theorem and the Gibbard-Satterthwaite theorem (Gibbard, 1973, and Satterthwaite, 1975) are the two most classical examples of impossibility results, and both are predicated on social choice rules defined for the universal domain of preference profiles: they hold for functions that must take values for any possible combination of agent's

V. Torra et al. (Eds.): MDAI 2013, LNAI 8234, pp. 11–24, 2013.

preferences over alternatives. By contrast, when social choice rules are defined in the restricted domain of single-peaked preferences, then the impossibilities turn into possibilities, and simple majority, among others, emerges as a fully satisfactory social welfare function generating strategy-proof social decisions as well (Black, 1948 and Moulin, 1980).

Single-peakedness is the best known, but not the only domain restriction under which the Arrowian aggregation difficulties can be overcome. Others are, for example, the restriction to single-crossing families of preferences, or to sets of profiles satisfying an intermediateness condition (Gans and Smart, 1996, Grantmond, 1978, Rothstein, 1990, Saporiti, 2009). Likewise, majority voting operates adequately within these domains, but other social choice rules may also be satisfactorily used (Austen-Smith and Banks, 1999). Characterizing the social choice rules that operate properly, in some well defined sense, under given domain restrictions, is a very fruitful approach to examine social welfare issues: Arrow's impossibility paves the way, but then positive characterization results follow. Similarly, the Gibbard-Satterthwaite theorem establishes that one may not expect to find nontrivial strategy-proof social choice rules operating on universal domains, but opens the door to different possibility results. We know the form of all the social choice rules that are strategy-proof when agent's preferences are single-peaked in different contexts (Moulin, 1980; Sprumont, 1991), but also of those that meet this condition under separable preferences (Barberà, Sonnenschein, and Zhou, 1991, Barberà, Gul, and Stacchetti, 1993, Barberà, Massó, and Neme, 1997, 1999, Barberà, Massó, and Serizawa, 1998, Serizawa, 1996), single-plateaued preferences (Berga, 1998), single-dipped (Barberà, Berga, and Moreno, 2012b,c; and Manjunath, 2013), for example.

The purpose of this expositional paper is to introduce the reader to three families of domains that we have identified in recent work as being sufficient (and close to necessary) to guarantee that social choice rules defined on them can satisfy a variety of desirable properties. In the spirit of our previous remarks, we shall present three puzzles in social choice, and show that their solution depends on the domain of preference profiles for which our relevant social choice rules are defined.

Here are the questions we want to address:

1. Is there a common root to the conditions of single-peakedness, single-crossing and intermediateness? It is known that voting equilibria under qualified majority rules can be guaranteed within these three domain restrictions and that, in addition, these equilibria are of similar form. We shall present the domain of top monotonic profiles that contains all three, and for which the existence of voting equilibria with essentially the same traits is still guaranteed.
2. What is the connection between individual and group strategy-proofness? It is known that for some domain restrictions, both conditions become equivalent, while in others individual strategy-proofness is a strictly weaker requirement. We shall present a domain condition called sequential inclusion where equivalence is implied, and that is “almost necessary” for the equivalence to hold.

3. When can strategy-proofness be guaranteed by the sole satisfaction of two simple and natural conditions of monotonicity and invariance? We shall exhibit a "connectedness" condition defining what we call intertwined domains and that ensures strategy-proofness thanks to these two conditions alone, while proving that in other cases it may be necessary to add further and less natural requirements.

Since the main purpose of this paper is expositional, we shall provide the results without their proofs, and refer for those to the original papers. After a brief section with general definitions, we devote one section to each of the puzzles that we just stated. In each one of them we try to motivate our question, describe the classes of domains for which we can provide definite answers and offer an example of alternative domains where social choice rules would fail to meet our requirements.

2 The Setup and Some Definitions

Let A be a set of *alternatives and* $N = \{1, ..., n\}$ be a finite set of *agents.* Let $\mathcal{R}$ be the set of all preorders (complete, reflexive, and transitive binary relations) on A and $\mathcal{R}_i \subseteq \mathcal{R}$ be *the set of admissible preferences for agent* $i \in N$. Denote by $\mathcal{P} \subseteq \mathcal{R}$ the set of all antisymmetric preorders. We denote by $R_i \in \mathcal{R}_i$ an admissible preference relation and let as usual, P_i and I_i be the strict and the indifference part of R_i, respectively. When all the admissible preferences for individual i are strict, we will use the notation $\mathcal{P}_i$, instead of the general expression $\mathcal{R}_i$. A *preference profile,* denoted by $R = (R_1, ..., R_n)$, is an element of $\times_{i \in N} \mathcal{R}_i$. For $C \subseteq N$ we will write the profile $R = (R_C, R_{N \backslash C}) \in \times_{i \in S} \mathcal{R}_i$ when we want to stress the role of coalition C. Then the subprofiles $R_C \in \times_{i \in C} \mathcal{R}_i$ and $R_{N \backslash C} \in \times_{i \in S \backslash C} \mathcal{R}_i$ denote the preferences of agents in C and in $N \backslash C$, respectively.

For any $R_i \in \mathcal{R}_i$ and $x \in A$, define *the lower contour set of* R_i *at* x as $L(R_i, x) = \{y \in A : xR_iy\}$. Similarly, the strict lower contour set at x is $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$.

For any $R_i \in \mathcal{R}_i$ and $B \subseteq A$, we denote by $t(R_i, B)$ *the set of maximal elements of* R_i *on* B. That is, $t(R_i, B) = \{x \in B : xR_iy \text{ for all } y \in B\}$. We call $t(R_i, B)$ the top of i in B or the peak on B when it is a singleton.

For any $x, y \in A$ and $R \in \times_{i \in N} \mathcal{R}_i$, let $P(x, y; R) \equiv \{i \in N : xP_iy\}$ and $R(x, y; R) \equiv \{i \in N : xR_iy\}$. That is, the set of agents who strictly (respectively, weakly) prefer x to y according to their individual preferences in R.

A *social choice function* is a function $f : \times_{i \in N} \mathcal{R}_i \to A$. Let A_f denote the range of the social choice function f. We say that f is *onto* if $A_f = A$.

We are interested in social choice functions that are nonmanipulable, either by a single agent or by a coalition of agents. We first define what we mean by a manipulation and then we introduce the well known concepts of *strategy-proofness* and *group strategy-proofness.*

Definition 1. *A social choice function* f *is group manipulable on* $\times_{i \in N} \mathcal{R}_i$ *at* $R \in \times_{i \in N} \mathcal{R}_i$ *if there exists a coalition* $C \subseteq N$ *and* $R'_C \in \times_{i \in C} \mathcal{R}_i$ *(*$R'_i \neq R_i$ *for any* $i \in C$*) such that* $f(R'_C, R_{N \backslash C}) P_i f(R)$ *for all* $i \in C$*. We say that* f *is* *individually manipulable if there exists a possible manipulation where coalition C is a singleton.*

Definition 2. *A social choice function f is group strategy-proof on $\times_{i \in N} \mathcal{R}_i$ if f is not group manipulable for any $R \in \times_{i \in N} \mathcal{R}_i$. Similarly, f is strategy-proof if it is not individually manipulable.*

Next we introduce the notions of a preference aggregation rule, voting rule and voting equilibrium. We follow closely Austen-Smith and Banks (1999) since our results will extend those that they present in Chapter 4 of their book.

Definition 3. *A preference aggregation rule is a map, $F : \times_{i \in N} \mathcal{R}_i \to \mathcal{B}$, where $\mathcal{B}$ denotes the set of all reflexive and complete binary relations on A. We denote by R_F the image of profile R under preference aggregation rule F.*

Definition 4. *Given any two profiles $R, R' \in \times_{i \in N} \mathcal{R}_i$ and $x, y \in A$, a preference aggregation rule F is:*
(1) neutral if and only if $[\forall a, b \in A,\ P(x,y;R) = P(a,b;R')$ and $P(y,x;R) = P(b,a;R')]$ imply $xR_F y$ if and only if $aR_F b$;
(2) monotonic if and only if $[P(x,y;R) \subseteq P(x,y;R'),\ R(x,y;R) \subseteq R(x,y;R')$ and $xP_F y]$ imply $xP'_F y$.

A neutral preference aggregation rule treats all alternatives equally when making pairwise comparisons. Monotonicity implies that if x is socially preferred to y and then some people change their preferences so that the support for x does not decrease, while the support for y does not increase, then x must be still socially preferred at the new profile.

One can always associate to each preference aggregation rule a family of ordered pairs of coalitions that represent the ability of different groups of agents in determining the social preference relation.

Definition 5. *The decisive structure associated with a preference aggregation rule F, denoted by $\mathcal{D}(F)$, is a family of ordered pairs of coalitions $(S, W) \subseteq N \times N$ such that $(S, W) \in \mathcal{D}(F) \Leftrightarrow S \subseteq W$ and $\forall x, y \in A$, $\forall R \in \times_{i \in N} \mathcal{R}_i$, $[xP_i y\ \forall i \in S$ and $xR_i y\ \forall i \in W \to xP_F y]$.*

We now notice that we could have started to define our preference aggregation rule by first providing a family of ordered pairs of coalitions.

Definition 6. *A set $\mathcal{D} \subset 2^N \times 2^N$ is*
(1) monotonic if $(S, W) \in \mathcal{D}$, $S \subseteq S' \subseteq W'$ and $S \subseteq W \subseteq W'$ imply $(S', W') \in \mathcal{D}$
(2) proper if $(S, W) \in \mathcal{D}$, $S' \subseteq N \backslash W$ and $W' \subseteq N \backslash S$ imply $(S', W') \notin \mathcal{D}$.

Definition 7. *Given a proper set D, the preference aggregation rule induced by $\mathcal{D}$, denoted $F_{\mathcal{D}}$, is defined as $\forall x, y \in A$, $xP_{F_{\mathcal{D}}} y \Leftrightarrow [\exists (S, W) \in \mathcal{D} : xP_i y\ \forall i \in S$ and $xR_i y\ \forall i \in W]$[1].*

[1] Notice that the requirement that $\mathcal{D}$ is proper guarantees that $f_{\mathcal{D}}$ is well defined.

It is useful to state the connections between preference aggregation rules and decisive structures, because one is closer to the language of social choice and the other is closer to that of public economics and political economy. More precisely, one can ask when it is the case that a decisive structure and a preference aggregation rule can be used interchangeably as being the primitives. This will happen when the decisive structure associated with F induces F again. Austen-Smith and Banks (1999) define voting rules as those preference aggregation rules that have this property, and provide a characterization for them.

Definition 8. *A preference aggregation rule F is a voting rule if $F = F_{\mathcal{D}(F)}$.*

Proposition 1. *A preference aggregation rule is a voting rule iff it is neutral and monotonic.*

In this survey we concentrate on the study of voting rules and their equilibria, which we now define.

Definition 9. *Let F be a preference aggregation rule and $R \in \times_{i \in N} \mathcal{R}_i$. The core of F at R, $C_F(R, S)$ is the set of maximal elements in $S \subseteq A$ under the binary relation R_F. Elements in the core of a voting rule will be called voting equilibria.*

3 Top Monotonicity

In this section we provide a condition on preference profiles, that we call top monotonicity. This condition, when satisfied by all profiles in a domain, guarantees that voting functions on that domain generate games with a non empty core. Moreover, these core elements, or voting equilibria, are generalized medians in the distribution of preferences for the agents. Furthermore, the classical domains of single-peaked, single-plateaued, single crossing or intermediate (order restricted) profiles are all included within this larger domain.

Voting rules are included among the methods that will fail to satisfy Arrow's theorem, when defined on the universal domain. When restricted to operate on the classical domains that we mention, they produce voting equilibria that are, in addition, nicely expressed as the medians of the distribution of voter's best elements. Our result unifies these possibility results by showing that, although the classical domains are different from each other, they all share one basic feature: they all satisfy our condition of top monotonicity. Moreover, this fact is sufficient to understand why the equilibria under these different restrictions also share the common structural fact of being closely linked to medians.

This section summarizes results that were first stated and proven in Barberà and Moreno (2011) where the authors propose a new condition on preference profiles over one-dimensional alternatives, called top monotonicity. And where they prove that top monotonicity can be viewed as the common root of a bunch of classical domain restrictions, which had been perceived in the literature as rather different and unrelated to each other.

We additionally assume that individual preferences are continuous binary relations on A. Let us introduce some notation: For each preference profile R, let $A(R)$ be the family of sets containing A itself, and also all triples of distinct alternatives where each alternative is top on A for some agent $k \in N$ according to R.

Now we present top monotonicity.

Definition 10. *A preference profile R is top monotonic iff there exists a linear order $>$ of the set of the alternatives, such that*
(1) $t(R_i, A)$ is a finite union of closed intervals for all $i \in N$, and
(2) For all $S \in A(R)$, for all $i, j \in N$, all $x \in t(R_i, S)$, all $y \in t(R_j, S)$, and any $z \in S$

$$[z < y < x \text{ or } z > y > x] \rightarrow \begin{array}{c} yR_iz \text{ if } z \in t(R_i, S) \cup t(R_j, S) \\ \text{and} \\ yP_iz \text{ if } z \notin t(R_i, S) \cup t(R_j, S). \end{array}$$

When convenient, we'll say that a preference profile is top monotonic relative to $>$.

We can begin by comparing top monotonicity with single-peakedness and single-plateauedness to see that it represents a significant weakening of these conditions. Single-peakedness requires each agent to have a unique maximal element. Moreover, it must be true for any agent that any alternative y to the right (left) of its peak is strictly preferred to any other that is further to the right (left) of it. In particular, this implies that no agent is indifferent between two alternative on the same side of its peak. Hence, indifference classes may consist of at most two alternatives (one to the right and one to the left of the agent's peak).

In contrast, our definition of top monotonicity allows for individual agents to have nontrivial indifference classes, both in and out of the top. In that respect, it allows for many more indifferences than single-plateaued preferences do. Most importantly, top monotonicity relaxes the requirement imposed on the ranking of two alternatives lying on the same side of the agent's top. Under our preference condition, this requirement is only effective for triples where the alternative that is closest to the top of the agent is itself a top element for some other agent. Moreover, the implication is only in weak terms when the alternative involved in the comparison is top for one or for both agents.

A similar, although less direct comparison can be made between top monotonicity and intermediateness or order restriction. The original conditions involve comparisons between pairs of alternatives, regardless of their positions in the ranking of agents. Top monotonicity is also a strict weakening of these requirements, involving the comparison of only a limited number of pairs.

We now state a first result showing that top monotonicity is a common root for a lot of typical preferences restrictions, as it is implied by any of them.

Theorem 1. *(see Theorem 1 in Barberà and Moreno, 2011) If a preference profile is either single-peaked, single-plateaued, single crossing or order restricted, then it also satisfies top monotonicity.*

We now show that top monotonicity guarantees the existence of voting equilibria under any voting rule, and that these will be closely connected to an extended notion of the median voter. Before stating this second result of the paper, we introduce some additional notation, and we propose an extension of the notion of median.

Let $>$ be a linear order of the set of alternatives and R be a preference profile. For any $z \in A$, we define the following three sets

$$N_{\{z\}} = \{j \in N : z \in t_j(A)\},$$

$$N_{\{z\}^-} = \{k \in N : z > x \text{ for all } x \in t_k(A)\},$$

and

$$N_{\{z\}^+} = \{h \in N : z < x \text{ for all } x \in t_h(A)\}.$$

We remark that when R is top monotonic relative to $>$, and z is in the top of some agent i, then $N_{\{z\}} \neq \emptyset$ and the three sets $(N_{\{z\}^-}, N_{\{z\}}, N_{\{z\}^+})$ constitute a partition of the set of voters N. Indeed, $N_{\{z\}}$ contains all voters, including i, for whom z is in the top. $N_{\{z\}^-}$ (resp. $N_{\{z\}^+}$) contains all voters for which all top elements are to the left (resp. to the right) of z. Clearly, then, these three sets are disjoint. To prove that their union contains all elements of N, suppose not. For some agent l, z should not be in $l's$ top, while some alternatives x and y, one to the right and one to the left of z, should belong to the top of l. But then, by top monotonicity we would have zR_lx and also zR_ly. Since x and y are both top for l, so is z, a contradiction[2].

Let n, $n_{\{z\}}$, $n_{\{z\}^-}$, and $n_{\{z\}^+}$ be the cardinalities of N, $N_{\{z\}^-}$, $N_{\{z\}}$ and $N_{\{z\}^+}$, respectively. From the remark above, we know that if z is in the top of some agent, then $n_{\{z\}} + n_{\{z\}^-} + n_{\{z\}^+} = n$. The following definition will allow us to establish an analogue of the classical median voter result for the case of top monotonic profiles.

Definition 11. *Let F be a voting rule. An alternative z is a weak F-median top alternative in a top monotonic profile R relative to an order $>$ of the set of alternatives iff*
(1) z is a top alternative in R for some agent, and
(2) $(N_{\{z\}^-}, N_{\{z\}^-} \cup N_{\{z\}}) \notin \mathcal{D}(F)$ and $(N_{\{z\}^+}, N_{\{z\}} \cup N_{\{z\}^+}) \notin \mathcal{D}(F)$.[3]

[2] Notice that our definition of top monotonicity does not preclude the possibility that an agent's top might be non-connected relative to the order of $>$. Informally, what it demands is that, if an agent has two peaks with a valley in between, then no other agent's peak lies in that valley. In that sense also, our condition is less demanding than that of single plateaued, where we assumed that the tops are connected.

[3] When f is the majority rule we say that an alternative z is a weak median top alternative in a top monotonic profile $\succcurlyeq$ relative to an order $>$ of the set of alternatives iff
(1) z is a top alternative in $\succcurlyeq$ for some agent, and
(2) $n_{\{z\}^-} + n_{\{z\}} \geqslant n_{\{z\}^+}$ and $n_{\{z\}} + n_{\{z\}^+} \geqslant n_{\{z\}^-}$.

We will denote by $WM_F(R)$ the set of weak F-median top alternatives at that profile. We define m^- and m^+ as the lowest and the highest elements in this set according to the order $>$ at that profile.

Definition 12. *An alternative z is an extended weak F-median in a top monotonic profile R relative to an order $>$ of the set of alternatives iff $m^- \leqslant z \leqslant m^+$.*

It is not hard to prove that extended medians in our sense always exist. We will denote by $M_F(R)$ the set of extended weak F-median alternatives at that profile.

We can now state the following result.

Theorem 2. *(see Theorem 1 in Barberà and Moreno, 2011)*
(1) Let F be a voting rule. Whenever a profile of preferences R is top monotonic relative to some order $>$, $C_F(R)$ is not empty and $C_F(R) \subseteq M_F(R)$.
(2) If the profile of preferences R is peak monotonic, $WM_F(R) \subseteq C_F(R)$.

4 Sequential Inclusion

In this section we define a domain condition that we call sequential inclusion. Social choice functions defined on domains that meet this condition will have the property that individual and group strategy-proofness become equivalent. That is, all individual strategy-proof social choice functions on domains satisfying sequential inclusion will also be immune to group manipulation.

Our research was prompted by the observation that this equivalence does not only trivially hold under the universal domain, where only dictatorial social choice functions can be strategy-proof, but also for much more interesting domains, like those of single-peaked preferences. However, both conditions are not equivalent in other contexts, where non trivial strategy-proof social choice functions do exist and yet are subject to manipulation by groups. What we do is to provide an (almost) exact frontier between those cases when the domain of definition of a social choice function does guarantee the equivalence between both properties, and those where it does not.

This section summarizes results that were first stated and proven in Barberà, Berga, and Moreno (2010). Our focus will be on specific cases where it is possible to at least define satisfactory strategy-proof social choice functions. The main question addressed in our paper was what is needed then to hope for the stronger and much more reassuring property of group strategy-proofness to also hold?

We start by defining our condition on preference profiles, called sequential inclusion, and we establish the equivalence between individual and group strategy-proofness for social choice functions defined on domains satisfying that condition. Let us introduce some notation.

Definition 13. *Given a preference profile $R \in \times_{i \in N} \mathcal{R}_i$ and a pair of alternatives $y, z \in A$, we define a binary relation $\succsim (R; y, z)$ on $P(y, z; R)$ as follows:*

$$i \succsim (R; y, z) j \text{ if } L(R_i, z) \subseteq \overline{L}(R_j, y).$$

Note that the binary relation $\succsim$ must be reflexive but not necessarily complete. As usual, we can define the strict and the indifference binary relations associated to $\succsim$. Formally, $i \sim j$ if $L(R_i, z) \subseteq \overline{L}(R_j, y)$ and $L(R_j, z) \subseteq \overline{L}(R_i, y)$. We say that iPj if $L(R_i, z) \subseteq \overline{L}(R_j, y)$ and $\neg[L(R_j, z) \subseteq \overline{L}(R_i, y)]$.

We can now define our main condition.

Definition 14. *A preference profile $R \in \times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion if for any pair $y, z \in A$ the binary relation $\succsim (R; y, z)$ on $P(y, z; R)$ is complete and acyclic. A domain $\times_{i\in N}\mathcal{R}_i$ satisfies sequential inclusion if any preference profile in this domain satisfies it.*

Since sequential inclusion is a property on preference profiles, it follows that if a domain satisfies sequential inclusion each subdomain inherits the same property. Remarkably, this condition does not require domains to be large in size, contrary to others, like ”richness” (see Dasgupta, Hammond, and Maskin [5], Le Breton and Zaporozhets [12]) or our own condition of indirect sequential inclusion defined in Barberà, Berga, and Moreno (2010).

We now present our first main result.

Theorem 3. *(see Theorem 1 in Barberà, Berga, and Moreno, 2010) Let $\times_{i\in N}\mathcal{R}_i$ be a domain satisfying sequential inclusion. Then, any strategy-proof social choice function on $\times_{i\in N}\mathcal{R}_i$ is group strategy-proof.*

A surprising result for the case of three alternatives is that when there are at most three alternatives at stake, any strategy-proof social choice function is group strategy-proof: This is mainly due to the fact that in such framework any preference profile satisfies sequential inclusion (see Corollary 1 in Barberà, Berga, and Moreno, 2010).

In our paper, we also provide another condition, weaker than sequential inclusion and called indirect sequential inclusion, that still guarantees the equivalence between individual and group strategy-proofness (see Theorem 2 in Barberà, Berga, and Moreno, 2010). It is no longer a condition on individual profiles. Rather, it requires that, given a profile within the domain, some other profile, conveniently related to the first one, does indeed satisfy our previous requirement. That is why we say that profiles that meet our new condition satisfy indirect sequential inclusion. The new definition allows us to incorporate new and interesting domains into our list of those guaranteeing equivalence. The interested reader can check the more cumbersome concept of indirect sequential inclusion in Definition 8 in Barberà, Berga, and Moreno (2010).

It is also worth mentioning the partial necessity result we obtain in our work: sequential inclusion is almost necessary to guarantee that individual and group strategy-proofness become equivalent (see Theorem 4 in Barberà, Berga, and Moreno, 2010).

To finish this section we present a simple example that illustrates how our results would fail on alternative domains. We exhibit one domain, that of separable preferences, that violates both direct and indirect sequential inclusion. This is because we can find a social choice function that is strategy-proof but not group strategy-proof on the mentioned domain. In view of Theorem 4 this proves that indirect sequential inclusion fails (thus, also the direct version).

Example 1. Two candidates a and b may be elected to join a club.[4] Alternatives in this problem are sets of candidates: $A = \{\varnothing, a, b, \{a,b\}\}$.
Given a preference on sets, candidates are called good if they are better than the empty set, when chosen alone, and bad otherwise. Preferences are separable if adding a good candidate to any set makes the union better, and adding a bad one makes the union worse. The set of individual separable preferences is the same for each agent $i \in N$:

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8
$\varnothing$	$\varnothing$	a	a	b	b	$\{a,b\}$	$\{a,b\}$
a	b	$\varnothing$	$\{a,b\}$	$\varnothing$	$\{a,b\}$	a	b
b	a	$\{a,b\}$	$\varnothing$	$\{a,b\}$	$\varnothing$	b	a
$\{a,b\}$	$\{a,b\}$	b	b	a	a	$\varnothing$	$\varnothing$

Consider the social choice function, called voting by quota one: each agent declares her best set of objects and any object that is mentioned by some agent is selected.

This social choice function is clearly strategy-proof. Yet notice that for a profile where $R_1 = R^3$, $R_2 = R^5$ and for any other agent $R_i = R^1$ the outcome would be $\{a,b\}$, whereas agents 1 and 2 could vote for $\varnothing$ and get a preferred outcome. Thus, it is not group strategy-proof since any pair of agents can manipulate it.

5 Intertwined Domains

In this section we define and study two conditions on social choice functions that we find especially attractive: reshuffling invariance and monotonicity. Reshuffling invariance and monotonicity are always necessary for strategy-proofness, whatever the domain of definition of the functions, but need not be sufficient. Because of that, we ask ourselves the following question: can we identify domains of preferences having the property that, when functions are defined on these domains, then our conditions are equivalent to strategy-proofness? We answer this question in the positive. For those domains that we call intertwined, and for any possible social choice function defined on them, the equivalence holds.

For this study we consider the set of alternatives to be finite and, for sake of simplicity, we assume that agents' preferences are strict.

This section presents results originally obtained in Barberà, Berga, and Moreno (2012a).

---

[4] The example easily extends to any set of candidates: just take profiles as we have just defined and extend individual preferences such that the relative ordering among a, b, $\{a,b\}$ and the empty set are like in the above table and any other object is bad.

Definition 15. *A social choice function f satisfies* ***monotonicity*** *on $\times_{i\in N}\mathcal{P}_i$ if and only if for any $P \in \times_{i\in N}\mathcal{P}_i$ such that $f(P) = x$, and for any $P' \in \times_{i\in N}\mathcal{P}_i$ satisfying the following conditions*

(i) for any $i \in N$, for any $y \in A\backslash\{x\}$; $[xP_iy \Rightarrow xP_i'y]$, and
(ii) for any $i \in N$, for any $y, z \in A\backslash\{x\}$; $[yP_iz \Leftrightarrow yP_i'z]$.

then, $f(P') = x$.

In words: If an alternative x is chosen by a social choice function f at profile $(P_i, P_{N\backslash\{i\}})$, and P_i' is a new preference where x has improved its position then f must still choose x.

Definition 16. *Let $P_i \in \mathcal{P}_i$ and $x \in A$. We say that $P_i' \in \mathcal{P}_i$ is a x-*reshuffling of P_i *if $\overline{L}(P_i, x) = \overline{L}(P_i', x)$.*

In words: P_i' is a x-reshuffling of P_i if it results from keeping all alternatives that were worse than x and no other, as still being worse, though maybe in a different order.

Definition 17. *A social choice function f satisfies* ***reshuffling invariance*** *on $\times_{i\in N}\mathcal{P}_i$ if and only if for any $P \in \times_{i\in N}\mathcal{P}_i$ such that $f(P) = x$, and for any $(P_i', P_{N\backslash\{i\}}) \in \times_{i\in N}\mathcal{P}_i$ such that P_i' is a x-reshuffling of P_i, then $f(P_i', P_{N\backslash\{i\}}) = x$.*

In words: If an alternative x is chosen at a profile, x must be chosen at any other profile obtained from an x-reshuffling of agent i's preferences.

We now introduce our notion of intertwined domains. Whether a domain is intertwined or not will turn out to be crucial to determine whether the different conditions we are interested in may or may not be equivalent, when applied to social choice functions defined on such domains.

Before we provide a formal definition, let us describe the condition informally. For any $i \in N$, select any two (strict) preferences P_i and P_i', and any two alternatives x and y, where xP_iy (the relationship between the two in P' can be any). Suppose that there exists in our domain a third preference $\overline{P}_i$ such that one can transform P_i into $\overline{P}_i$, through a sequence of changes in the positions of alternatives, such that these changes, at each step, simply consist in lifting the position of y, or of reshufflings around y. Suppose that one can also transform P_i' into $\overline{P}_i$ through another sequence of the same type of transformations, this time with liftings of x and reshufflings around x. We will then say that P_i and P_i' are (x, y)-intertwined.

A domain of preferences will be intertwined if and only if any two of the preferences it contains are intertwined for any two alternatives.

Even more informally, we can say that an intertwined domain is one where one can travel from any pair of preferences to some intermediary preference just by lifting and reshuffling alternatives.

We now proceed to our formal definitions.

Definition 18. *Let $P_i, \overline{P}_i \in \mathcal{P}_i$ and $x \in A$. We say that $\overline{P}_i$ is a x-**direct transform of** P_i if either $\overline{P}_i$ is a x-reshuffling of P_i or $\overline{P}_i$ is a x-monotonic transformation of P_i.*

Definition 19. *Let $P_i, \overline{P}_i \in \mathcal{P}_i$ and $x \in A$. We say that $\overline{P}_i$ is a x-**transform of** P_i if there exist a sequence of preferences $P_1, P_2, ..., P_T$ such that $P_1 = P_i$, $P_T = \overline{P}_i$, and for any $t \in (1, T]$, each P_t is a x-direct transform of P_{t-1}.*

Definition 20. *Let $P_i, P'_i \in \mathcal{P}_i$, $x, y \in A$ where xP_iy. We say that P_i is (x, y)-**intertwined with** P'_i if there exists $\overline{P}_i \in \mathcal{P}_i$ such that $\overline{P}_i$ is both a y-transform of P_i and a x-transform of P'_i.*

Definition 21. *A set of individual preferences $\mathcal{P}_i$ is **intertwined** if for any $P_i \in \mathcal{P}_i$, for any $x, y \in A$ such that xP_iy, and any $P'_i \in \mathcal{P}_i$, P_i is (x, y)-intertwined with P'_i.*

Definition 22. *A domain $\times_{i \in N} \mathcal{P}_i$ is intertwined if for any agent i, $\mathcal{P}_i$ is intertwined.*

We are now ready to state our equivalence result.

Theorem 4. *(see Theorem 1 in Barberà and Moreno, 2012a) Any social choice function defined on an intertwined domain is strategy-proof if and only if it satisfies monotonicity and reshuffling invariance.*

As we already proved in Proposition 1 (1) in Barberà, Berga, and Moreno (2012a), let us remark that monotonicity and reshuffling invariance, our two independent conditions, are necessary for any social choice function defined on any domain to be strategy-proof. However, as we show in the following example our conditions are not always sufficient to guarantee strategy-proofness: there exist social choice functions satisfying both of them which are nevertheless manipulable; clearly the domain of preferences is crucial as Theorem 4 states.

Example 2. (borrowed from the proof of Proposition 1 (3) in Barberà, Berga, and Moreno, 2012a)

Consider the framework in Example 1

Our example refers to a social choice function defined on the domain of separable preferences for the case of two candidates, four alternatives and three voters:

Define the social choice function as the Borda count on A with tie breaking. Voters rank the four alternatives, and each alternative gets three points whenever a voter ranks it first, two when ranked second, one when third and none if last. The choice is the alternative with the highest sum of points, if unique. As for possible ties, notice that, in our example, when there is a tie for first position. there may be at most one voter for whom none of the tied alternatives is the best for him. If there is such an individual, the tie is broken in favor of that alternative that he prefers. Otherwise, the tie is broken according to a pre-determined order of alternatives, say $O : \{a, b\}, b, a, \varnothing$.

Notice that the only cases where the antecedents of reshuffling invariance and monotonicity would apply are those where we change a preference to another having the same top. Given that, it is easy to see that both conditions are respected in our example.

Yet, observe that the function is still manipulable. To see that, let $P = (P^1, P^6, P^7)$, $P' = (P^6, P^6, P^7)$. Then, $f(P) = b$ (b and $\{a, b\}$ have the same score and agent 1 breaks the tie) and $f(P') = \{a, b\}$ (b and $\{a, b\}$ have the same score but all agents have b or $\{a, b\}$ as best alternative, so we use O). Thus, agent 1 manipulates f at P' via P^1.

To finish this section two comments are in order. First, with strict preferences and under intertwined domains, our two conditions are not only equivalent to strategy-proofness but also to strong positive association (Muller and Satterthwaite, 1977). However, strong monotonicity (Moulin, 1988) is weaker than monotonicity and reshuffling invariance conditions (see Proposition 5 in Barberà, Berga, and Moreno, 2012a). Second, let us mention that, in general, intertwinedness and (indirect) sequential inclusion are independent. However, intertwinedness implies indirect sequential inclusion when the set of individual preferences are strict and equal for all agents (see Proposition 6 in Barberà, Berga, and Moreno, 2012a).

6 Concluding Remarks

Our main message is that every specific social choice problem deserves a careful analysis of domains on which we need to define the method to be used, since this may open the doors to attractive possibility results. We have exemplified this message by presenting three domains that we have found worth studying and hope that the readers find them useful for their further work.

Ackowledgements. Salvador Barberà gratefully acknowledges support from "Consolidated Group-C" ECO2008-04756 and FEDER, SGR2009-0419, and the Severo Ochoa Programme for Centres of Excellence in R&D, SEV-2011-0075. Dolors Berga acknowledges the support of the Spanish Ministry of Science and Innovation through grant ECO2010-16353, of Generalitat de Catalunya, through grant SGR2009-0189. Bernardo Moreno gratefully acknowledges financial support from Junta de Andalucia through grants SEJ4941 and SEJ-5980 and the Spanish Ministry of Science and Technology through grant ECO2011-29355.

References

Austen-Smith, D., Banks, J.S.: Positive Political Theory I. Collective Preference. The University of Michigan Press (1999)

Barberà, S., Berga, D., Moreno, B.: Individual versus group strategy-proofness: when do they coincide? J. Econ. Theory 145, 1648–1674 (2010)

Barberà, S., Berga, D., Moreno, B.: Two necessary conditions for strategy-proofness: on what domains are they also sufficient? Games Econ. Beh. 75, 490–509 (2012a)

Barberà, S., Berga, D., Moreno, B.: Domains, ranges and strategy-proofness: the case of single-dipped preferences. Soc. Choice Welfare 39(2-3), 335–352 (2012b)

Barberà, S., Berga, D., Moreno, B.: Group strategy-proof social choice functions with binary ranges and arbitrary domains: characterization results. Int. J. Game Theory 41(4), 791–808 (2012c)

Barberà, S., Gul, F., Stacchetti, E.: Generalized Median Voter Schemes and Committees. J. Econ. Theory 61, 262–289 (1993)

Barberà, S., Massó, J., Neme, A.: Voting under Constraints. J. Econ. Theory 76(2), 298–321 (1997)

Barberà, S., Massó, J., Neme, A.: Maximal domains of preferences preserving strategy-proofness for generalized median voter schemes. Soc. Choice Welfare 16(2), 321–336 (1999)

Barberà, S., Massó, J., Serizawa, S.: Strategy-proof Voting on Compact Ranges. Games Econ. Beh. 25(2), 272–291 (1998)

Barberà, S., Moreno, B.: Top monotonicity: a common root for single peakedness, single crossing and the median voter result. Games Econ. Beh. 73, 345–359 (2011)

Barberà, S., Sonnenschein, H., Zhou, L.: Voting by committees. Econometrica 59, 595–609 (1991)

Berga, D.: Strategy-Proofness and Single-Plateaued Preferences. Math. Soc. Sci. 35(2), 105–120 (1998)

Dasgupta, P., Hammond, P., Maskin, E.: The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility. Rev. Econ. Stud. 46, 185–216 (1979)

Gans, J.S., Smart, M.: Majority Voting with Single-Crossing Preferences. J. of Public Econ. 59, 219–237 (1996)

Grandmond, J.-M.: Intermediate Preferences and Majority Rule. Econometrica 46, 317–330 (1978)

Gibbard, A.: Manipulation of Voting Schemes: A General Result. Econometrica 41, 587–601 (1973)

Le Breton, M., Zaporozhets, V.: On the Equivalence of Coalitional and Individual Strategy-Proofness Properties. Soc. Choice Welfare 33, 287–309 (2009)

Manjunath, V.: Efficient and strategy-proof social choice when preferences are single-dipped. Forthcoming in Int. J. Game Theory (2013)

Moulin, H.: On Strategy-proofness and Single Peakedness. Public Choice 35, 437–455 (1980)

Moulin, H.: Axioms of cooperative decision making. Econometric Society Monograph (1988)

Muller, E., Satterthwaite, M.A.: On the Equivalence of Strong Positive Association and Strategy-Proofness. J. Econ. Theory 14, 412–418 (1977)

Rothstein, P.: Order Restricted Preferences and Majority Rule. Soc. Choice Welfare 7, 331–342 (1990)

Saporiti, A.: Strategy-proofness and single-crossing. Theoretical Economics 4, 127–163 (2009)

Satterthwaite, M.: Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions. J. Econ. Theory 10, 187–217 (1975)

Serizawa, S.: Power of Voters and Domain of Preferences Where Voting by Committees Is Strategy-Proof. J. Econ. Theory 67, 599–608 (1995)

Sprumont, Y.: Strategyproof Collective Choice in Economic and Political Environments. Can. J. Econ. 28(1), 68–107 (1995)